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Canonical Projectors for Linear Differential Algebraic Equations

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Abstract—This paper aims at improving the decoupling projector chain approach for DAEs to provide complete decouplings. These special canonical projectors are shown to form exactly the spectral projection related to the matrix coefficient pair given by the DAE. Further, they prove their value when describing the constraint manifolds of the DAE.

Keywords—Matrix pencil, Spectral projection, Differential algebraic equation, Descriptor system.

1. INTRODUCTION

Linear constant coefficient differential algebraic equations (DAEs)

$$Ax'(t) + Bx(t) = q(t)$$

are best understood. Well-known classic tools to investigate them are the transformation into Kronecker normal form and the decoupling by means of Drazin inverses and spectral projectors. For a respective survey, we refer to [1]. No doubt, both tools are very smart. However, of course they also have certain disadvantages. In particular, there are no sufficiently good ideas on appropriate generalizations for variable coefficient linear DAEs and nonlinear ones, respectively.

A different way (e.g., [2]) to deal with DAEs consists in decoupling them by means of a special matrix and projector chain. Fortunately, the matrix and projector chain approach applies also in the case of general variable coefficient equations (e.g., [3]). Further, there is some first experience to use those decouplings via linearizations for lower index nonlinear problems. In particular, März [4] proposes a special choice of the projector chain leading to a simple but complete decoupling of index 2 linear constant coefficient DAEs. Those projectors are called canonical ones. They are approved to be a useful tool, e.g., for stating local solvability, asymptotical stability (cf. [4]).

Actually, some of the most important questions in discussing DAEs seem to be whether the DAE induces a vector field on a manifold and how the state manifold can be described in terms of the original DAE (cf. [5,6]). Also from this point of view the canonical projector chain has proved its value (e.g., [7]).

On this background, the present paper aims at improving the projector chain even for the constant coefficient equations. In Section 2, it is shown that the projectors P_j may be chosen such that they are canonical; i.e., the related decoupling becomes complete. Section 3 describes both the induced vector field and the state manifold in terms of the matrix chain and the canonical projectors $P_0, \dots, P_{\mu-1}$. More precisely, the state manifold is proved to be $\operatorname{im} P_0 \cdots P_{\mu-1}$. Finally, comparing with the spectral projector decoupling (Section 4) shows that $P_0 \cdots P_{\mu-1}$ represents in fact the projector onto the subspace corresponding to the finite eigenvalues of the matrix pair $\{A, B\}$ along its infinite eigenspace.

Notice that the matrix chain and the canonical projectors are available in practice. In particular, they may also be used for testing numerically the index, for obtaining consistent initial values, etc. It seems that there remain a lot of further good possibilities to be exploited.

2. CANONICAL PROJECTORS AND COMPLETE DECOUPLING OF DAES

Linear constant coefficient DAEs have been studied for many years. They are well understood. Given a coefficient pair $\{A, B\}$, $A, B \in L(\mathbb{R}^m)$ which forms a regular matrix pencil $\lambda A + B$, the DAE

$$Ax'(t) + Bx(t) = q(t) \tag{2.1}$$

may be reduced to its Kronecker normal form

$$u'(t) + Wu(t) = r(t) \tag{2.2a}$$

$$Jv'(t) + v(t) = s(t) \tag{2.2b}$$

by transforming the pencil

$$\lambda EAF + EBF = \lambda \begin{pmatrix} I & \\ & J \end{pmatrix} + \begin{pmatrix} W & \\ & I \end{pmatrix}.$$

J is a nilpotent Jordan chain matrix. Its Riesz-index μ ($J^\mu = 0$, $J^{\mu-1} \neq 0$) is said to be the Kronecker index of both the pencil and (the coefficient pair $\{A, B\}$ of) the DAE.

Of course, the transforms E, F are not known explicitly which causes this procedure to be not very useful in view of practical computations and when asking for generalizations, too. This is why one tries to decouple (2.1) by means of certain projectors that are better available. One such possible decoupling is realized by means of the following matrix and projector chain. Put $A_0 := A$, $B_0 := B$, further

$$A_{j+1} := A_j + B_j Q_j, \quad B_{j+1} := B_j P_j, \qquad j \geq 0, \tag{2.3}$$

whereby $Q_j \in L(\mathbb{R}^m)$ denotes a projector onto the nullspace $\ker A_j$, $P_j := I - Q_j$.

The sequence $A_0, A_1, \dots$ is known to become stationary, i.e., $A_{\mu+j} = A_\mu$, $j \geq 0$, supposing the pencil $\lambda A + B$ is regular [2].

Choosing somewhat special projectors Q_j within (2.3), we obtain an appropriate tool for decoupling the DAE (2.1).

THEOREM 2.1. *Given a regular index μ DAE (2.1).*

(i) *Then $A_0, \dots, A_{\mu-1}$ are singular but A_μ is nonsingular.*
(ii) $\dim \ker A_{j+1} = \dim S_j \cap \ker A_j$, *where* $S_j := \{z \in \mathbb{R}^m : B_j z \in \operatorname{im} A_j\}$, $j \geq 0$.
(iii) *The projectors $Q_0, \dots, Q_{\mu-1}$ may be chosen such that*

$$Q_j Q_i = 0, \qquad \text{for } i < j. \tag{2.4}$$

The proof is referred to [2, Section 1].

Due to (2.4), all of the products of projectors arising in

$$I = P_{\mu-1} + Q_{\mu-1} = P_0 \cdots P_{\mu-1} + Q_0 P_1 \cdots P_{\mu-1} + \cdots + Q_{\mu-2} P_{\mu-1} + Q_{\mu-1} \tag{2.5}$$

or

$$I = P_0 + Q_0 = P_0 \cdots P_{\mu-1} + P_0 \cdots P_{\mu-2} Q_{\mu-1} + \cdots + P_0 Q_1 + Q_0 \tag{2.6}$$

are projectors again. Further, (2.5), (2.6) represent appropriate decompositions for decoupling the DAE. In particular, assertion (ii) yields

$$S_{\mu-1} \cap \ker A_{\mu-1} = \{0\},$$

which allows us to choose $Q_{\mu-1}$ to project onto $\ker A_{\mu-1}$ along $S_{\mu-1}$. Then [8, Lemma A.14] the identity

$$Q_{\mu-1} = Q_{\mu-1} A_\mu^{-1} B_{\mu-1} \tag{2.7}$$

holds true. The $Q_{\mu-1}$ is called a canonical projector.

Return to the DAE (2.1) and scale it by A_μ^{-1} to

$$P_{\mu-1} \cdots P_0 x'(t) + A_\mu^{-1} B x(t) = A_\mu^{-1} q(t), \tag{2.8}$$

taking into account that (2.3) implies

$$A = A_\mu P_{\mu-1} \cdots P_0.$$

Assume (2.4) to be valid. Then we decompose (2.8) by (2.5), that is, we multiply (2.8) by $P_0 \cdots P_{\mu-1}, Q_0 P_1 \cdots P_{\mu-1}, \dots, Q_{\mu-2} \cdot P_{\mu-1}$ and $Q_{\mu-1}$, respectively. This gives, after some technical computations, the system

$$P_0 \cdots P_{\mu-1} x'(t) + P_0 \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{\mu-1} x(t) = P_0 \cdots P_{\mu-1} A_\mu^{-1} q(t), \tag{2.9a}$$

$$\begin{aligned} &-\begin{pmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_{\mu-2} \end{pmatrix} \begin{pmatrix} I & P_1 & P_2 & \cdots & P_1 \cdots P_{\mu-2} \\ & I & P_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots \quad \ddots & P_{k-2} \\ & & & & I \end{pmatrix} \begin{pmatrix} Q_1 x'(t) \\ Q_2 x'(t) \\ \vdots \\ Q_{\mu-1} x'(t) \end{pmatrix} \\ &+ \begin{pmatrix} Q_0 x(t) \\ Q_1 x(t) \\ \vdots \\ Q_{\mu-2}(t) \end{pmatrix} + \begin{pmatrix} \tilde{Q}_0 P_0 \cdots P_{\mu-1} x(t) \\ \tilde{Q}_1 P_0 \cdots P_{\mu-1} x(t) \\ \vdots \\ \tilde{Q}_{k-2} P_0 \cdots P_{\mu-1} x(t) \end{pmatrix} = \begin{pmatrix} Q_0 P_1 \cdots P_{\mu-1} A_\mu^{-1} q(t) \\ Q_1 P_2 \cdots P_{\mu-1} A_\mu^{-1} q(t) \\ \vdots \\ Q_{\mu-2} P_{\mu-1} A_\mu^{-1} q(t) \end{pmatrix}, \end{aligned} \tag{2.9b}$$

$$Q_{\mu-1} x(t) + \tilde{Q}_{\mu-1} P_0 \cdots P_{\mu-1} x(t) = Q_{\mu-1} A_\mu^{-1} q(t). \tag{2.9c}$$

Here we have denoted shortly $\tilde{Q}_j := Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B_j$, $j = 0, \dots, \mu-2$, $\tilde{Q}_{\mu-1} = Q_{\mu-1} A_\mu^{-1} B_{\mu-1}$. The first equation (2.9a) obviously represents the inherent in (2.1) regular ODE for the component $P_0 \cdots P_{\mu-1} x$ (the state variable). This part corresponds to (2.2a). The other equations are to determine the components $Q_{\mu-1}x, \dots, Q_0 x$. In particular, $Q_{\mu-1}x$ is given by (2.9c). The part (2.9b) appears for $\mu \geq 2$ only. However, if $\mu \geq 2$, then $Q_{\mu-2}x$ is obtained by differentiating the expression for $Q_{k-1}x$ and so on.

As mentioned before, we can choose $Q_{\mu-1}$ to be canonical, i.e., to satisfy (2.7). If we do so, we have $Q_{\mu-1} = \tilde{Q}_{\mu-1}$, hence, $\tilde{Q}_{\mu-1} P_0 \cdots P_{\mu-1} = Q_{\mu-1} P_0 \cdots P_{\mu-1} = Q_{\mu-1} P_{\mu-1} = 0$, and (2.9c) simplifies to

$$Q_{\mu-1} x(t) = Q_{\mu-1} A_\mu^{-1} q(t). \tag{2.10}$$

Thus, there is no more coupling between (2.10) and (2.9a). However, there remains a bother coupling of (2.9b) and (2.9a). Is it possible to drop the related coupling terms in (2.9b) in a similar way by special projectors also?

First of all, $\tilde{Q}_j$ is easily checked to be also a projection matrix onto $\ker A_j$, $0 \leq j \leq \mu - 1$. So we are asking for projectors satisfying the relations

$$Q_j = \tilde{Q}_j = Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B_j, \qquad j = 0, \dots, \mu - 1. \tag{2.11}$$

Those projectors are called canonical ones. Clearly, (2.11) yields $\tilde{Q}_j P_0 \cdots P_{\mu-1} = Q_j P_0 \cdots P_{\mu-1} = Q_j P_j \cdots P_{\mu-1} = 0$ for $j = 0, \dots, \mu - 1$. Hence, if we choose canonical projectors in advance, then the coupling terms in (2.9b) disappear. But, may we choose those canonical projectors, do they exist? The answer is given by the next theorem. Fortunately, those projectors do exist.

EXAMPLE 1. Many authors are interested in semi-explicit equations where the derivative free equations are split off, say

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \tag{2.12}$$

Supposing A_{11} is a regular block and A, B form a regular index 1 pencil, we have

$$Q_0 = \begin{pmatrix} -A_{11}^{-1} A_{12} G_{22}^{-1} B_{21} & -A_{11}^{-1} A_{12} G_{22}^{-1} B_{22} \\ G_{22}^{-1} B_{21} & G_{22}^{-1} B_{22} \end{pmatrix}, \tag{2.13}$$

whereby $G_{22} := B_{22} - B_{21} A_{11}^{-1} A_{12}$ is a regular block due to the index 1 property. Recall that Q_0 projects onto

$$\ker A = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : A_{11} u + A_{12} v = 0 \right\}$$

along

$$S_0 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : B_{21} u + B_{22} v = 0 \right\}.$$

If, additionally, $A_{11} = I$, $A_{12} = 0$, then Q_0 simplifies to

$$Q_0 = \begin{pmatrix} 0 & 0 \\ B_{22}^{-1} B_{21} & I \end{pmatrix}. \tag{2.14}$$

THEOREM 2.2. *Given a regular index μ matrix pair $\{A, B\}$. Then there are projectors $Q_0, \dots, Q_{\mu-1} \in L(\mathbb{R}^m)$ such that*

$$\begin{aligned} Q_0 &= Q_0 P_1 \cdots P_{\mu-1} A_\mu^{-1} B, \\ Q_j &= Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{j-1}, \qquad j = 1, \dots, \mu - 2 \\ Q_{\mu-1} &= Q_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{\mu-2}. \end{aligned}$$

The proof is given in the Appendix. At this place, it should be mentioned that Theorem 2.2 generalizes [8, Lemma A.14], which applies in the index 1 case.

3. DESCRIBING THE CONSTRAINT MANIFOLDS OF DAES BY MEANS OF CANONICAL PROJECTORS

In general, DAEs are closely related to vector fields on manifolds (cf. [5]), but both the related vector field and the manifold are not known explicitly. Thus we try to describe them in terms that are available in practice. Consider the homogeneous equation

$$Ax'(t) + Bx(t) = 0, \tag{3.1}$$

where $\{A, B\}$ forms a regular index μ pencil. Surely, if the Kronecker normal form (2.2) of this DAE were given and k denoted the dimension of the u-component, we would see immediately that the v-component vanishes identically and

$$z = -F \begin{pmatrix} W & \\ & 0 \end{pmatrix} F^{-1} x = x \in M := \left\{ F \begin{pmatrix} u \\ 0 \end{pmatrix} : u \in \mathbb{R}^k \right\} \tag{3.2}$$

represent the vector field on the state manifold generated by (3.1). However, both the transforms E, F and the system matrix W are not well available in practice.

Let us return to the original DAE (3.1). Introduce the subspaces (cf. [6])

$$\mathcal{G}_0 := \{(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Az + Bx = 0\},$$
$$M_0 := \{x \in \mathbb{R}^m : Bx \in \operatorname{im} A\} = \pi_x \mathcal{G}_0.$$

Thereby π_x is the projection onto the first component of $\mathbb{R}^m \times \mathbb{R}^m$. If the C^1-function $x(\cdot) : J \to \mathbb{R}^m$ solves the DAE (3.1), then the relations

$$(x(t), x'(t)) \in \mathcal{G}_0, \qquad t \in J,$$
$$x(t) \in M_0, \qquad t \in J,$$

are valid, further

$$(x(t), x'(t)) \in TM_0 = M_0 \times M_0, \qquad t \in J,$$

where TM_0 denotes the tangent bundle of M_0, thus

$$(x(t), x'(t)) \in \mathcal{G}_1 := \mathcal{G}_0 \cap TM_0, \qquad t \in J,$$
$$x(t) \in M_1 := \pi_x \mathcal{G}_1, \qquad t \in J.$$

Next we describe the set

$$\mathcal{G}_1 = \{(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Az + Bx = 0,\ Bz \in \operatorname{im} A\}$$

and its projection M_1 in more detail. For these aims, we decouple the linear systems

$$Ax + Bx = 0 \quad \text{and} \quad Ay + Bz = 0$$

by means of the canonical projectors provided by Theorem 2.2. This leads to

$$P_0 \cdots P_{\mu-1} z + P_0 \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{\mu-1} x = 0, \tag{3.3a}$$

$$\begin{pmatrix} Q_0 x \\ \vdots \\ Q_{\mu-2} x \end{pmatrix} = - \begin{pmatrix} Q_0 & & \\ & \ddots & \\ & & Q_{\mu-2} \end{pmatrix} \begin{pmatrix} I & P_1 \ldots & P_1 \cdots P_{\mu-2} \\ & \ddots & \vdots \\ & & P_{k-2} \\ & & I \end{pmatrix} \begin{pmatrix} Q_1 z \\ \vdots \\ Q_{\mu-1} z \end{pmatrix}, \tag{3.3b}$$

$$Q_{\mu-1} x = 0, \tag{3.3c}$$

as well as

$$P_0 \cdots P_{\mu-1} y + P_0 \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{\mu-1} z = 0, \tag{3.4a}$$

$$\begin{pmatrix} Q_0 z \\ \vdots \\ Q_{\mu-2} z \end{pmatrix} = \begin{pmatrix} Q_0 & & \\ & \ddots & \\ & & Q_{\mu-2} \end{pmatrix} \begin{pmatrix} I & P_1 \cdots & P_1 \cdots P_{\mu-2} \\ \ddots & \ddots & \\ & & P_{\mu-2} \\ & & I \end{pmatrix} \begin{pmatrix} Q_1 y \\ \vdots \\ Q_{\mu-1} y \end{pmatrix}, \tag{3.4b}$$

$$Q_{\mu-1} z = 0. \tag{3.4c}$$

Obviously now we have due to (3.3)

$$M_0 = \{x \in \mathbb{R}^m : Q_{\mu-1} x = 0,\ Q_{\mu-j} x \in \operatorname{im} D_{\mu-j},\ j = 2, \ldots, \mu\},$$

where

$$D_{\mu-j} := Q_{\mu-j}\left(Q_{\mu-j+1} + P_{\mu-j+1}Q_{\mu-j+2} + \cdots + P_{\mu-j+1}\cdots P_{\mu-2}Q_{\mu-1}\right)$$
$$= Q_{\mu-j}P_{\mu-j+1}\cdots P_{\mu-1}A_\mu^{-1}A.$$

Further, (3.3), (3.4) lead to

$$\mathcal{G}_1 = \Big\{(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Q_{\mu-1}x = 0,\ Q_{\mu-1}z = 0,\ Q_{\mu-2}x = 0,\ Q_{\mu-j}x = D_{\mu-j}z,\ j = 3,\dots,\mu,$$
$$Q_{\mu-j}z \in \operatorname{im} D_{\mu-j},\ j = 2,\dots,\mu,\ P_0\cdots P_{\mu-1}z = -P_0\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}x\Big\},$$
$$M_1 = \pi_1\mathcal{G}_1 = \{x \in \mathbb{R}^m : Q_{\mu-1}x = 0,\ Q_{\mu-2}x = 0,\ Q_{\mu-j}x \in \operatorname{im} D_{\mu-j},\ j = 3,\dots,\mu\}.$$

In the index 1 case, that is, for $\mu = 1$, we arrive at

$$M_0 = \{x \in \mathbb{R}^m : Q_0x = 0\} = \operatorname{im} P_0,$$
$$\mathcal{G}_1 = \{(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Q_0x = 0,\ Q_0z = 0,\ P_0z = -P_0A_1^{-1}BP_0x\}$$

and we get $M_1 = M_0$. For each $x \in M_0$, the set $\mathcal{G}_1$ provides a unique vector $z \in \mathbb{R}^m$. Therefore, the DAE (3.1) generates the vector field

$$z = P_0z = -P_0A_1^{-1}BP_0x, \qquad x \in M_0 = \operatorname{im} P_0.$$

Next we turn to the case $\mu = 2$. Then we have

$$\mathcal{G}_1 = \Big\{(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Q_1x = 0,\ Q_1z = 0,\ Q_0x = 0,\ Q_0z \in \operatorname{im} Q_0Q_1,$$
$$P_0P_1z = -P_0P_1A_2^{-1}BP_0P_1x\Big\}.$$
$$M_1 = \{x \in \mathbb{R}^m : Q_1x = 0, Q_0x = 0\} = \operatorname{im} P_0P_1$$

is a proper subset of

$$M_0 = \{x \in \mathbb{R}^m : Q_1x = 0,\ Q_0x \in \operatorname{im} Q_0Q_1\}.$$

Specifying (3.3) and (3.4) for $\mu = 2$ yields

$$\begin{aligned} P_0P_1z + P_0P_1A_2^{-1}BP_0P_1x &= 0, & P_0P_1y + P_0P_1A_2^{-1}BP_0P_1z &= 0,\\ Q_0x &= Q_0Q_1z, & Q_0z &= Q_0Q_1y,\\ Q_1x &= 0, & Q_1z &= 0. \end{aligned}$$

Consequently, we find

$$TM_1 = \{(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Q_1x = 0,\ Q_0x = 0,\ Q_1z = 0, Q_0z = 0\},$$
$$\mathcal{G}_2 := \mathcal{G}_1 \cap TM_1 = \Big\{(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Q_1x = 0,\ Q_0x = 0,\ Q_1z = 0,\ Q_0z = 0,$$
$$P_0P_1z = -P_0P_1A_2^{-1}BP_0P_1x\Big\},$$
$$M_2 := \pi_x\mathcal{G}_2 = \{x \in \mathbb{R}^m : Q_0x = 0,\ Q_1x = 0\} = M_1.$$

In $\mathcal{G}_2$, there is exactly one z for each $x \in M_1$. Hence, we obtain the vector field

$$z = P_0P_1z = -P_0P_1A_2^{-1}BP_0P_1x, \qquad x \in M_1 = \operatorname{im} P_0P_1.$$

Now, returning to the general case, we derive

$$\begin{aligned}\mathcal{G}_2 = \mathcal{G}_1 \cap TM_1 = \Big\{&(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Q_{\mu-1}x = 0,\ Q_{\mu-2}x = 0,\ Q_{\mu-1}z = 0,\ Q_{\mu-2}z = 0,\\ &Q_{\mu-j}x = D_{\mu-j}z,\ j = 3,\dots,\mu,\ Q_{\mu-j}z \in \operatorname{im} D_{\mu-j},\ j = 3,\dots,\mu,\ P_0\cdots P_{\mu-1}z\\ &= -P_0\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}x\Big\}\\ = \Big\{&(x,z) \in \mathbb{R}^m \times \mathbb{R}^m : Q_{\mu-1}x = 0,\ Q_{\mu-2}x = 0,\ Q_{\mu-3}x = 0,\ Q_{\mu-1}z = 0,\ Q_{\mu-2}z = 0,\\ &Q_{\mu-j}x = D_{\mu-j}z,\ j = 4,\dots,\mu,\ Q_{\mu-j}z \in \operatorname{im} D_{\mu-j},\ j = 3,\dots,\mu,\ P_0\cdots P_{\mu-1}z\\ &= -P_0\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}x\Big\},\end{aligned}$$

further

$$M_2 := \pi_x\mathcal{G}_2 = \{x \in \mathbb{R}^m : Q_{\mu-1}x = 0,\ Q_{\mu-2}x = 0,\ Q_{\mu-3}x = 0, Q_{\mu-j}x \in \operatorname{im} D_{\mu-j},\ j = 4,\dots,\mu\}.$$

At this point, we know that the following assertion can be verified.

THEOREM 3.1. *The given index μ DAE (3.1) generates the vector field*

$$z = -P_0\cdots P_{\mu-1}A_\mu^{-1}Bx, \qquad x \in M_{\mu-1} = \operatorname{im} P_0\cdots P_{\mu-1}.$$

For the constraint manifolds M_i, $i = 0,\dots,\mu$, it holds that

$$M_i = \{x \in \mathbb{R}^m : Q_{\mu-j}x = 0,\ j = 1,\dots,i+1, Q_{\mu-j}x \in \operatorname{im} D_{\mu-j},\ j = i+2,\dots,\mu\},$$

further $M_0 \supset M_1 \supset \cdots \supset M_{\mu-1} = M_\mu$.

Comparing this result with the Kronecker normal form mentioned above (cf. (2.2), (3.2)), we find the next assertion.

COROLLARY 3.2. *For an index μ matrix pair $\{A, B\}$ and its Kronecker normal form, it holds that*

$$\operatorname{im} P_0\cdots P_{\mu-1} = \left\{F\begin{pmatrix}u\\0\end{pmatrix} : u \in \mathbb{R}^k\right\}, \qquad P_0\cdots P_{\mu-1}A_\mu^{-1}B = F\begin{pmatrix}W & \\ & 0\end{pmatrix}F^{-1}.$$

4. COMPARING WITH SPECTRAL PROJECTIONS

Another different way to decouple a linear constant coefficient DAE is described in [9]. It goes back to spectral projections and Drazin inverses.

Let G^D denote the Drazin inverse of a given matrix $G \in L(\mathbb{R}^m)$. Then G^DG is known to be the so-called spectral projector which projects $\mathbb{R}^m$ onto $\operatorname{im} G^\mu$ along $\ker G^\mu$. Here μ denotes the index of G, that is, the smallest integer k for which $\ker G^k = \ker G^{k+1}$ is true. The decomposition

$$\operatorname{im} G^\mu \oplus \ker G^\mu = \mathbb{R}^m$$

is known to be valid then.

Consider the regular index μ matrix pair $\{A, B\}$. Choose $c \in \mathbb{R}$ such that $\det(cA + B) \neq 0$ and introduce

$$\hat{A} := (cA + B)^{-1}A, \qquad \hat{B} := (cA + B)^{-1}B.$$

Then we turn to the DAE

$$\hat{A}x'(t) + \hat{B}x(t) = 0, \tag{4.1}$$

i.e., the equation (3.1) scaled by the nonsingular matrix $(cA + B)^{-1}$. Next we multiply (4.1) by $\hat{A}^D$ and $\hat{B}^D(I - \hat{A}^D\hat{A})$, respectively. Carrying out a few technical calculations (e.g., [8–10]), we derive the system

$$\tilde{u}'(t) + \hat{A}^D\hat{B}\tilde{u}(t) = 0, \tag{4.2a}$$

$$\hat{B}^D\hat{A}\tilde{v}'(t) + \tilde{v}(t) = 0. \tag{4.2b}$$

System (4.2) is equivalent to (4.1) via the decomposition

$$x = Rx + (I - R)x =: \tilde{u} + \tilde{v},$$

where $R := \hat{A}^D\hat{A}$ denotes the projector onto $\operatorname{im}\hat{A}^\mu$ along $\ker\hat{A}^\mu$.

Taking into account that $\hat{A}, \hat{B}$ commute, further

$$\left(\hat{B}^D\hat{A}\right)^\mu (I - R) = \left(\hat{B}^D\right)^\mu \hat{A}^\mu(I - R) = 0,$$

we know the solution component $\tilde{v}(t) = (I - R)\tilde{v}(t)$ to vanish identically. In this context, the vector field induced by the DAE (3.1) is now described as

$$z = -\hat{A}^D\hat{B}x, \qquad x \in \operatorname{im} R = \operatorname{im}\hat{A}^\mu. \tag{4.3}$$

As a consequence, we obtain the next two assertions.

THEOREM 4.1. *Given regular index μ pair $\{A, B\}$. Then, for the canonical projectors provided by Theorem 2.2, it holds that*

$$M_{\mu-1} = \operatorname{im} P_0 \cdots P_{\mu-1} = \operatorname{im}\hat{A}^D\hat{A} = \operatorname{im}\hat{A}^\mu. \tag{4.4}$$

COROLLARY 4.2. *For $A \in L(\mathbb{R}^m)$, $\mu := \operatorname{ind} A$, it holds that*

$$\operatorname{im} P_0 \cdots P_{\mu-1} = \operatorname{im} AA^D = \operatorname{im} A^\mu,$$

with projectors $P_0, \dots, P_{\mu-1}$ provided by Theorem 2.2 for the pair $\{A, I\}$.

Notice that Theorem 4.1 generalizes [8, Theorem A.15], where the respective index 1 relation $S_0 = \operatorname{im} P_0 = \operatorname{im}\hat{A}$ is given.

The subspace $\operatorname{im}\hat{A}^\mu$ corresponds to the so-called relative finite eigenvalues of the pair $\{A, B\}$, that is, to the eigenvalues of the block $-W$ in the Kronecker form (2.2). On the other hand, $\ker\hat{A}^\mu$ is said to be the relative infinite eigenspace of $\{A, B\}$ (cf. [1]).

We close this section by discussing to some extent the system (cf. [11])

$$p' - v = 0, \tag{4.5a}$$

$$v' + Kp + Dv + C^\top\lambda = f, \tag{4.5b}$$

$$0 = Cp - h, \tag{4.5c}$$

which may be understood as a (linearized) equation of motion for a multibody system with rigid or elastic bodies and massless interconnections. p, v are position and velocity coordinates, λ denotes the generalized constraint force. K, D stand for the stiffness and damping matrices, C for the constraint matrix, f, h for the system and the kinematric excitations. C is supposed to have full rank. System (4.5) is well known to represent an index 3 DAE.

In [11], the related ODE $x'(t) = -\hat{A}^D\hat{B}x(t)$ is called the "Drazin ODE of the DAE." Further, its coefficient matrix $\hat{A}^D\hat{B}$ as well as the spectral projector $R = \hat{A}^D\hat{A}$ are explicitly derived as

$$-\hat{A}^D\hat{B} = \begin{bmatrix} (I-H)DH & I-H & 0 \\ -(I-H)\mathcal{A}_{21} & -(I-H)D(I-H) & 0 \\ F^\top\mathcal{A}_{31} & F^\top\{D(I-H)D-K\}(I-H) & 0 \end{bmatrix}, \tag{4.6}$$

$$R = \begin{bmatrix} I-H & 0 & 0 \\ (I-H)DH & I-H & 0 \\ -F^\top\{K(I-H)+D(I-H)DH\} & -F^\top D(I-H) & 0 \end{bmatrix}, \tag{4.7}$$

where

$$\begin{aligned} F &:= C^\top\left(CC^\top\right)^{-1}, \qquad H := FC, \\ \mathcal{A}_{21} &:= K(I-H)+D(I-H)DH, \\ \mathcal{A}_{31} &:= \{-K+D(I-H)D\}(I-H)DH + D(I-H)K(I-H). \end{aligned}$$

On the other hand, the matrix chain for

$$A_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 & -I & 0 \\ K & D & C^\top \\ C & 0 & 0 \end{bmatrix}, \qquad Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

yields (cf. [12])

$$Q_2 = \begin{bmatrix} H & 0 & 0 \\ H-(I-H)DH & 0 & 0 \\ -F^\top(I+DH) & 0 & 0 \end{bmatrix}.$$

Further, we have

$$S_2 = \{x \in \mathbb{R}^m : B_2x \in \operatorname{im} A_2\} = \left\{ \begin{pmatrix} p \\ v \\ \lambda \end{pmatrix} \in \mathbb{R}^m : Cp = 0 \right\},$$

and Q_2 projects onto $\ker A_2$ along S_2. $Q_2 = Q_2A_3^{-1}BP_0P_1$, $Q_1 = Q_1P_2A_3^{-1}BP_0$ are satisfied. Next we form

$$\bar{Q}_0 := Q_0P_1P_2A_3^{-1}B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F^\top\{K(I-H)-D(I-H)(I-D)H\} & F^\top D(I-H) & I \end{bmatrix}.$$

Forming a new matrix chain with $\bar{Q}_0$ as initial projector instead of Q_0, we obtain with $\bar{Q}_0$, $\bar{Q}_1 = Q_1$, $\bar{Q}_2 = Q_2$ canonical projectors due to Lemma A.2. With the product

$$\bar{P}_0\bar{P}_1\bar{P}_2 = \begin{bmatrix} I-H & 0 & 0 \\ (I-H) & I-H & 0 \\ -F^\top\{K(I-H)+D(I-H)DH\} & -F^\top D(I-H) & 0 \end{bmatrix}, \tag{4.8}$$

we obtain exactly the same projector as given in (4.7), thus

$$R = \bar{P}_0\bar{P}_1\bar{P}_2, \qquad \text{further } \bar{M} = \bar{P}_0\bar{P}_1\bar{P}_2\bar{A}_3^{-1}B.$$

Comparing with (4.7), (4.6), we verify

$$\bar{P}_0\bar{P}_1\bar{P}_2 = R = \hat{A}^D\hat{A}, \qquad M = \hat{A}^D\hat{B} \tag{4.9}$$

for the system (4.5), and hence, the matrix chain provides precisely the spectral projector with $\bar{P}_0\bar{P}_1\bar{P}_2$.

Naturally the question arises whether $P_0 \cdots P_{\mu-1}$ and $R = \hat{A}^D\hat{A}$ are identical in general.

THEOREM 4.3. *For any regular index μ pair $\{A, B\}$ and the canonical projectors given by Theorem 2.2, it holds that $P_0 \cdots P_{\mu-1} = R$, i.e., $P_0 \cdots P_{\mu-1}$ projects onto* $\operatorname{im} \hat{A}^\mu$ *along* $\ker \hat{A}^\mu$.

PROOF. The special construction of the canonical projectors leads to

$$M_{\mu-1} = \ker A_0 \oplus \ker A \oplus \cdots \oplus \ker A_{\mu-1}.$$

Taking into account formula (4.4), it remains to show that

$$\ker A_0 \oplus \cdots \oplus \ker A_{\mu-1} = \ker \hat{A}^\mu. \tag{4.10}$$

Denote for short $N_i := \ker A_i$, $i = 0, \ldots, \mu - 1$. Next we prove the inclusions

$$\ker \hat{A}^k \subseteq N_0 \oplus \cdots \oplus N_{k-1}, \qquad k = 1, \ldots, \mu \tag{4.11}$$

by means of simple induction. For $k = 1$, $\ker \hat{A} = \ker A = N_0$ is given trivially. Let $x \in \ker \hat{A}^2$, $x = \alpha_0 x_0 + \alpha_1 x_1$ with $\hat{A}x_0 = 0$, $\hat{A}x_1 = x_0$, or equivalently

$$Ax_0 = 0, \quad Ax_1 = (cA + B)x_0 = Bx_0. \tag{4.12}$$

Since $x_0 = Q_0x_0$, the second relation of (4.12) may be rewritten as $(A + BQ_0)(P_0x_1 - Q_0x_0) = 0$. Thus, $P_0x_1 - Q_0x_0 \in \ker A_1 = N_1$,

$$P_0x_1 - Q_0x_0 = Q_1 (P_0x_1 - Q_0x_0) = Q_1x_1,$$

hence $x_1 = Q_0x_1 + P_0x_1 = Q_0x_1 + Q_0x_0 + Q_1x$ belongs to $N_0 \oplus N_1$, thus $x \in N_0 \oplus N_1$.

Suppose (4.11) to be valid for $k \leq \ell$ and consider $x \in \ker \hat{A}^{\ell+1}$,

$$x = \alpha_0x_0 + \cdots + \alpha_{\ell-1}x_{\ell-1} + \alpha_\ell x_\ell$$

with $\hat{A}x_0 = 0$, $\hat{A}x_1 = x_0, \ldots, \hat{A}x_\ell = x_{\ell-1}$, or equivalently

$$\begin{aligned} &Ax_0 = 0, Ax_1 = Bx_0, Ax_2 = cBx_0 + Bx_1, \ldots \\ &Ax_\ell = c^{\ell-1}Bx_0 + \cdots + cBx_{\ell-2} + Bx_{\ell-1}. \end{aligned} \tag{4.13}$$

Because of $x_0 \in N_0, \ldots, x_{\ell-1} \in N_0 \oplus \cdots \oplus N_{\ell-1}$, the expression $\mathcal{B} := c^{\ell-1}Bx_0 + \cdots + Bx_{\ell-1}$ may be reformulated as

$$\mathcal{B} = BQ_0\tilde{x}_0 + BP_0Q_1\tilde{x}_1 + \cdots + BP_0 \cdots P_{\ell-2}Q_{\ell-1}\tilde{x}_{\ell-1}.$$

Recall further that $A_\ell = A + BQ_0 + \cdots + BP_0 \cdots P_{\ell-2}Q_{\ell-1}$, $A_\ell P_{\ell-1} \cdots P_0 = A$, $A_\ell Q_0 = BQ_0$, $A_\ell Q_i = BP_0 \cdots P_{i-1}Q_i$, $i = 1, \ldots, \ell$. Hence from (4.13), it follows that

$$A_\ell (P_{\ell-1} \cdots P_0x_\ell - Q_0\tilde{x}_0 - \cdots - Q_{\ell-1}\tilde{x}_{\ell-1}) = 0,$$

or

$$P_{\ell-1} \cdots P_0x_\ell - Q_0\tilde{x}_0 - \cdots - Q_{\ell-1}\tilde{x}_{\ell-1} = Q_\ell x_\ell. \tag{4.14}$$

Finally, decomposing

$$x_\ell = P_0x_\ell + Q_0x_\ell = P_1P_0x_\ell + Q_1x_\ell + Q_0x_\ell = \cdots = P_{\ell-1} \cdots P_0x_\ell + Q_{\ell-1}x_\ell + \cdots + Q_0x_\ell$$

and taking into account that (4.14) implies

$$P_{\ell-1}\cdots P_0 x_\ell \in N_0 \oplus \cdots \oplus N_{\ell-1} \oplus N_\ell,$$

we learn that x_ℓ belongs to $N_0 \oplus \cdots \oplus N_\ell$, but then x does so, too. Thus we are done with (4.11).

Due to (4.11), $\ker \hat{A}^\mu$ is a subspace of $N_0 \oplus \cdots \oplus N_{\mu-1}$. For reasons of dimensions,

$$\ker \hat{A}^\mu = N_0 \oplus \cdots \oplus N_{\mu-1}$$

has to be true then indeed. ∎

COROLLARY 4.4. *Given $A \in L(\mathbb{R}^m)$, $\mu = \operatorname{ind} A$ and projectors $P_0, \dots, P_{\mu-1}$ provided by Theorem 2.2 for the pair $\{A, I\}$. Then*

$$P_0 \cdots P_{\mu-1} = AA^D. \tag{4.15}$$

Possibly, formula (4.15) offers an acceptable way for computing the spectral projector AA^D numerically. However, there is no experience on this up to now.

APPENDIX
PROOF OF THEOREM 2.2

Given a regular index μ matrix pencil $\{A, B\}$, $A, B \in L(\mathbb{R}^m)$. Starting with any projector $Q_0 \in L(\mathbb{R}^m)$ onto the nullspace of $A_0 := A$, we form the matrix chain

$$\begin{aligned} &A_{j+1} := A_j + B_j Q_j, \quad B_{j+1} := B_j P_j, \\ &\quad P_j := I - Q_j, \qquad Q_j \in L(\mathbb{R}^m), \quad Q_j^2 = Q_j, \quad \operatorname{im} Q_j = \ker A_j, \quad j \ge 0, \quad B_0 := B. \end{aligned}$$

Due to [2, Section 1], the matrices $A_0, \dots, A_{\mu-1}$ are singular but A_μ is nonsingular. Further, the projectors $Q_1, \dots, Q_{\mu-1}$ may be chosen such that

$$Q_j Q_i = 0, \qquad \text{for } j > i$$

and $Q_{\mu-1}$ projects onto $\ker A_{\mu-1}$ along the subspace

$$S_{\mu-1} := \{z \in \mathbb{R}^m : B_{\mu-1} z \in \operatorname{im} A_{\mu-1}\}.$$

Then, $Q_{\mu-1} = Q_{\mu-1} A_\mu^{-1} B_{\mu-1}$ holds true, i.e., $Q_{\mu-1}$ is canonical.

Next we consider

$$\bar{Q}_0 := Q_0 P_1 \cdots P_{\mu-1} A_\mu^{-1} B_0.$$

Because of

$$B_0 Q_0 = A_\mu Q_0, \quad \bar{Q}_0 Q_0 = Q_0,$$

it turns out that $\bar{Q}_0$ represents another projector onto $\ker A$.

Now, starting with $\bar{A}_0 := A$, $\bar{B}_0 := B$, and $\bar{Q}_0$, we proceed as above and generate a new matrix chain $\bar{A}_j$, $\bar{B}_j$, $j = 1, \dots, \mu$. However, this time we realize a special choice also of the projectors $\bar{Q}_1, \dots, \bar{Q}_{\mu-1}$.

LEMMA A.1.

$$\begin{aligned} \bar{A}_j &= A_j F_j, \\ F_j &:= I + \bar{Q}_0 P_0 + \bar{Q}_1 P_1 + \cdots + \bar{Q}_{j-1} P_{j-1}, \\ \bar{Q}_j &:= Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B_j \text{ projects onto } \ker \bar{A}_j, \qquad j = 1, \dots, \mu-1. \end{aligned}$$

PROOF. We have

$$\begin{aligned} \bar{A}_1 &= A + B\bar{Q}_0 = A + BQ_0\bar{Q}_0 = A + BQ_0 - BQ_0\bar{P}_0 \\ &= A_1 + A_1 \bar{Q}_0 P_0 = A_1 F_1. \end{aligned}$$

The matrix $F_1 = I + \bar{Q}_0 P_0$ is nonsingular, its inverse is $F_1^{-1} = I - \bar{Q}_0 P_0$. Let $\bar{Q}_1 := Q_1 P_2 \cdots P_{\mu-1} A_\mu^{-1} B_1$. Is it an appropriate projector? Since $B_1 Q_1 = A_\mu Q_1$, we have $\bar{Q}_1 Q_1 = Q_1$, $\bar{Q}_1^2 = \bar{Q}_1$. Hence, $\bar{Q}_1$ is a projector onto $\ker A_1$. In consequence,

$$\tilde{Q}_1 := F_1^{-1} \bar{Q}_1 F_1$$

represents a projector onto $\ker \bar{A}_1$. Trivially, $\bar{Q}_1 F_1 = \bar{Q}_1$, further $\bar{Q}_0 P_0 Q_1 = Q_0 P_1 \cdots P_{\mu-1} A_\mu^{-1} B P_0 Q_1 = 0$, thus $\bar{Q}_0 P_0 \bar{Q}_1 = 0$, and finally

$$\tilde{Q}_1 = \bar{Q}_1.$$

In fact, $\bar{Q}_1$ is the right projector. It has the following useful properties:

$$\bar{Q}_1 \bar{Q}_0 = 0, \qquad \bar{Q}_1 Q_0 = 0, \qquad Q_1 \bar{P}_1 = -\bar{Q}_1 P_1, \qquad \bar{Q}_0 \bar{Q}_1 = Q_0 \bar{Q}_1.$$

Now, let $\bar{A}_j = A_j F_j$, $\bar{Q}_j = Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B_j$, and $\bar{P}_0 \cdot \bar{P}_{j-1} \bar{Q}_j = P_0 \cdots P_{j-1} \bar{Q}_j$, $j = 1, \ldots, k \le \mu - 1$, be valid. We show that these relations become true also for $j = k+1$.

Taking into account that

$$\begin{aligned} A_k F_k = A_{k+1} P_k F_k = A_{k+1} \left(P_k + \bar{Q}_0 P_0 + \cdots + \bar{Q}_{k-1} P_{k-1}\right), \\ B P_0 \cdots P_{k-1} \bar{Q}_k = B P_0 \cdots P_{k-1} Q_k \bar{Q}_k = A_{k+1} Q_k \bar{Q}_k, \end{aligned}$$

we compute

$$\begin{aligned} \bar{A}_{k+1} &= \bar{A}_k + \bar{B}_k \bar{Q}_k + A_k F_k + B \bar{P}_0 \cdots \bar{P}_{k-1} \bar{Q}_k \\ &= A_{k+1} \left(P_k + \bar{Q}_0 P_0 + \cdots + \bar{Q}_{k-1} P_{k-1} + Q_k \bar{Q}_k\right) \\ &= A_{k+1} \left(I + \bar{Q}_0 P_0 + \cdots + \bar{Q}_{k-1} P_{k-1} - Q_k \bar{P}_k\right) = A_{k+1} F_{k+1}. \end{aligned}$$

Next we consider

$$\bar{Q}_{k+1} := Q_{k+1} P_{k+2} \cdots P_{\mu-1} A_\mu^{-1} B_{k+1}.$$

Clearly, $\bar{Q}_{k+1}^2 = \bar{Q}_{k+1}$, $\bar{Q}_{k+1} Q_{k+1} = Q_{k+1}$ is true since $B_{k+1} Q_{k+1} = A_\mu Q_{k+1}$. Therefore, $\bar{Q}_{k+1}$ is a projector onto $\ker A_{k+1}$, but then

$$\tilde{Q}_{k+1} := F_{k+1}^{-1} \bar{Q}_{k+1} F_{k+1}$$

represents a projector onto $\ker \bar{A}_{k+1}$. Note that $F_{k+1}^{-1} = I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_k P_k$. From $P_0 \cdots P_k Q_i = 0$, for $i = 0, \ldots, k$, it follows that

$$\bar{Q}_{k+1} F_{k+1} = \bar{Q}_{k+1}.$$

On the other hand, for $j = 0, \ldots, k$, it holds that

$$\begin{aligned} \bar{Q}_j P_j \bar{Q}_{k+1} &= \bar{Q}_j P_j Q_{k+1} \bar{Q}_{k+1} \\ &= Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{j-1} P_j Q_{k+1} \bar{Q}_{k+1} \\ &= Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_k Q_{k+1} \bar{Q}_{k+1} \\ &= 0, \end{aligned}$$

thus $F_{k+1}^{-1} \bar{Q}_{k+1} = \bar{Q}_{k+1}$, and we find the projectors $\bar{Q}_{k+1}$ and $\tilde{Q}_{k+1}$ to be the same. It remains to check whether

$$\bar{P}_0 \cdots \bar{P}_k \bar{Q}_{k+1} = P_0 \cdots P_k \bar{Q}_{k+1}$$

is also valid. Namely, we have

$$\begin{aligned}
\bar{P}_0 \cdots \bar{P}_k \bar{Q}_{k+1} &= \left(I - \bar{Q}_0 - \bar{P}_0\bar{Q}_1 - \cdots - \bar{P}_0 \cdots \bar{P}_{k-1}\bar{Q}_k\right) \bar{Q}_{k+1}, \\
&= \left(I - \bar{Q}_0 - P_0\bar{Q}_1 - \cdots - P_0 \cdots P_{k-1}\bar{Q}_k\right) \bar{Q}_{k+1}, \\
\bar{Q}_j \bar{Q}_{k+1} &= Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{j-1} \bar{Q}_{k+1} \\
&= Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{j-1} \left(P_j + Q_j\right) \bar{Q}_{k+1} \\
&= Q_j \bar{Q}_{k+1} + Q_j P_{j+1} \cdots P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_k Q_{k+1} \bar{Q}_{k+1} \\
&= Q_j \bar{Q}_{k+1}, \qquad j = 0, \ldots, k,
\end{aligned}$$

and thus

$$\begin{aligned}
\bar{P}_0 \cdots \bar{P}_k \bar{Q}_{k+1} &= \left(I - Q_0 - \cdots - P_0 \cdots P_{k-1} Q_k\right) \bar{Q}_{k+1} \\
&= P_0 \cdots P_k \bar{Q}_{k+1}.
\end{aligned}$$

∎

Since $Q_{\mu-1}$ projects onto $\ker A_{\mu-1}$ along the subspace $S_{\mu-1}$, it holds that (cf. [8, Lemma A.14])

$$Q_{\mu-1} = Q_{\mu-1} A_\mu^{-1} B_{\mu-1}.$$

On the other hand, Lemma A.1 leads to

$$\bar{Q}_{\mu-1} := Q_{\mu-1} A_\mu^{-1} B_{\mu-1} = Q_{\mu-1},$$

and hence, the projectors $Q_{\mu-1}, \bar{Q}_{\mu-1}$ are identical. In particular, this means

$$\ker A_{\mu-1} = \ker \bar{A}_{\mu-1}, S_{\mu-1} = \bar{S}_{\mu-1},$$

where we simply denoted

$$\bar{S}_{\mu-1} := \left\{ z \in \mathbb{R}^m : \bar{B}_{\mu-1} z \in \operatorname{im} \bar{A}_{\mu-1} \right\}.$$

Due to [8, Lemma A.14], the relation for canonical projectors

$$\bar{Q}_{\mu-1} = \bar{Q}_{\mu-1} \bar{A}_\mu^{-1} \bar{B}_{\mu-1}$$

is given for our new (marked by "$-$") projectors. Next we show $\bar{Q}_{\mu-2}$ to be also canonical; i.e.,

$$\bar{Q}_{\mu-2} = \bar{Q}_{\mu-2} \bar{P}_{\mu-1} \bar{A}_\mu^{-1} \bar{B}_{\mu-2}.$$

Namely, we have $P_{\mu-1} = \bar{P}_{\mu-1}$, $\bar{A}_\mu = A_\mu F_\mu$, $F_\mu = F_{\mu-1} = I + \bar{Q}_0 P_0 + \cdots + \bar{Q}_{\mu-1} = I + \bar{Q}_0 P_0 + \cdots + \bar{Q}_{\mu-2} P_{\mu-1}$, and

$$\begin{aligned}
\bar{Q}_{\mu-2} \bar{P}_{\mu-1} \bar{A}_\mu^{-1} \bar{B}_{\mu-2} &= \bar{Q}_{\mu-2} \bar{P}_{\mu-1} \bar{A}_\mu^{-1} B \bar{P}_0 \cdots \bar{P}_{\mu-3} = \bar{Q}_{\mu-2} \bar{P}_{\mu-1} \bar{A}_\mu^{-1} B \\
&= Q_{\mu-2} P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{\mu-3} P_{\mu-1} F_\mu^{-1} A_\mu^{-1} B \\
&= Q_{\mu-2} P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{\mu-3} \left(P_{\mu-2} + Q_{\mu-2}\right) P_{\mu-1} F_\mu^{-1} A_\mu^{-1} B \\
&= Q_{\mu-2} P_{\mu-1} A_\mu^{-1} B P_0 \cdots P_{\mu-1} F_\mu^{-1} A_\mu^{-1} B + Q_{\mu-2} P_{\mu-1} F_\mu^{-1} A_\mu^{-1} B.
\end{aligned}$$

Since

$$\begin{aligned}
P_0 \cdots P_{\mu-1} F_\mu^{-1} &= P_0 \cdots P_{\mu-1} \left(I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{\mu-2} P_{\mu-2}\right) \\
&= P_0 \cdots P_{\mu-1}, \\
Q_{\mu-2} P_{\mu-1} F_\mu^{-1} &= Q_{\mu-2} \left(P_{\mu-1} - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{\mu-2} P_{\mu-2}\right) \\
&= Q_{\mu-2} P_{\mu-1} - Q_{\mu-2} \bar{Q}_{\mu-2} P_{\mu-2} \\
&= Q_{\mu-2} P_{\mu-1} - \bar{Q}_{\mu-2} P_{\mu-2},
\end{aligned}$$

we obtain

$$\bar{Q}_{\mu-2}\bar{P}_{\mu-1}\bar{A}_\mu^{-1}\bar{B}_{\mu-2} = Q_{\mu-2}P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}A_\mu^{-1}B + Q_{\mu-2}P_{\mu-1}A_\mu^{-1}B - Q_{\mu-2}P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-2}A_\mu^{-1}B.$$

Adding the first and last term on the right-hand side, we derive

$$\begin{aligned}\bar{Q}_{\mu-2}\bar{P}_{\mu-1}\bar{A}_\mu^{-1}\bar{B}_{\mu-2} &= -Q_{\mu-2}P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-2}Q_{\mu-1}A_\mu^{-1}B + Q_{\mu-2}P_{\mu-1}A_\mu^{-1}B\\ &= Q_{\mu-2}P_{\mu-1}A_\mu^{-1}B\\ &= Q_{\mu-2}P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-3}.\end{aligned}$$

Thus we have arrived at

$$\bar{Q}_{\mu-2} := Q_{\mu-2}P_{\mu-1}A_\mu^{-1}B_{\mu-2} = \bar{Q}_{\mu-2}\bar{P}_{\mu-1}\bar{A}_\mu^{-1}\bar{B}_{\mu-2};$$

that means, $\bar{Q}_{\mu-2}$ is a canonical projector.

Recall what we have proved by now. In the original matrix chain, only the last projector $Q_{\mu-1}$ is canonical. (Clearly, if $\mu = 1$, we are done.) Restarting the procedure and applying Lemma A.1, we find another matrix chain (marked by "$\bar{\ }$") having already the two canonical projectors $\bar{Q}_{\mu-1} = Q_{\mu-1}$, $\bar{Q}_{\mu-2}$. This gives rise to formulating the following Lemma A.2.

Proving Lemma A.2, we verify Theorem 2.2 at the same time.

LEMMA A.2. *Let the projectors $Q_{\mu-1}, \dots, Q_k$ be canonical, i.e.,*

$$Q_j = Q_jP_{j+1}\cdots P_{\mu-1}A_\mu^{-1}B_j, \qquad j = \mu-1, \dots, k,$$

is valid. Then $\bar{Q}_j = Q_j$, $j = \mu-1, \dots, k$, and

$$\bar{Q}_{k-1} = \bar{Q}_{k-1}\bar{P}_k \cdot \bar{P}_{\mu-1}\bar{A}_\mu^{-1}\bar{B}_{k-1}.$$

PROOF. The equality of $\bar{Q}_j$ and Q_j, $j = \mu-1, \dots, k$, is simply guaranteed by the construction (see Lemma A.1). This simplifies the expression for F_μ to

$$F_\mu = I + \bar{Q}_0P_0 + \cdots + \bar{Q}_{k-1}P_{k-1}.$$

We derive

$$\begin{aligned}&\bar{Q}_{k-1}\bar{P}_k\cdots\bar{P}_{\mu-1}\bar{A}_\mu^{-1}\bar{B}_{k-1}\\ &= \bar{Q}_{k-1}\bar{P}_k\cdots\bar{P}_{\mu-1}\bar{A}_\mu^{-1}B\\ &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{k-2}\left(P_{k-1}+Q_{k-1}\right)P_k\cdots P_{\mu-1}F_\mu^{-1}A_\mu^{-1}B\\ &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}F_\mu^{-1}A_\mu^{-1}B + Q_{k-1}P_k\cdots P_{\mu-1}F_\mu^{-1}A_\mu^{-1}B\\ &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}A_\mu^{-1}B + \left(Q_{k-1}P_k\cdots P_{\mu-1} - \bar{Q}_{k-1}P_{k-1}\right)A_\mu^{-1}B.\end{aligned}$$

Because of

$$\begin{aligned}&Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}A_\mu^{-1}B - \bar{Q}_{k-1}P_{k-1}A_\mu^{-1}B\\ &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{\mu-1}A_\mu^{-1}B - Q_{k-1}P_k\cdots P_{\mu-1}\\ &\quad A_\mu^{-1}BP_0\cdots P_{k-2}\cdot P_{k-1}A_\mu^{-1}B\\ &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{k-1}\left(I - P_k\cdots P_{\mu-1}\right)A_\mu^{-1}B\\ &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}BP_0\cdots P_{k-1}\left(P_k\cdots P_{\mu-2}Q_{\mu-1} + \cdots + P_kQ_{k+1} + Q_k\right)A_\mu^{-1}B\\ &= 0,\end{aligned}$$

we conclude

$$\begin{aligned}\bar{Q}_{k-1}\bar{P}_k\cdots\bar{P}_{\mu-1}\bar{A}_\mu^{-1}\bar{B}_{k-1} &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}B\\ &= Q_{k-1}P_k\cdots P_{\mu-1}A_\mu^{-1}B_{k-1} = \bar{Q}_{k-1}.\end{aligned}$$

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